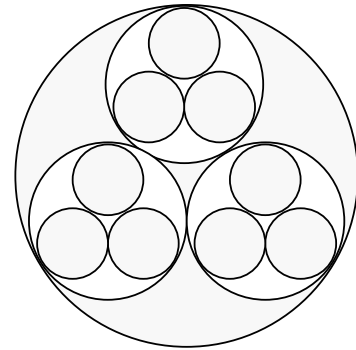


# Percolation: An Easy Example of Renormalization

Malcolm Forster, October 12, 2003.

Kenneth Wilson won the Nobel Prize in Physics in 1982 for applying renormalization group, which he learnt from quantum field theory (QFT), to problems in statistical physics—the induced magnetization of materials (ferromagnetism) and the evaporation and condensation of fluids (phase transitions). See Wilson (1983). The renormalization group got its name from its early applications in QFT. There, it appeared to be a rather ad hoc method of subtracting away unwanted infinities. The further allegation was that the procedure is so horrendously complicated that one cannot see the forest for the trees. The second allegation is justified in the applications that made it famous. But it is not true of the following example, which appears in Chowdhury and Stauffer (2000, 486-488).<sup>1</sup>



**Figure 1:** The smallest circles represent atoms. They are clustered into units of three, and so on, until we are left with one large circle, which defines the macroscopic boundary of a piece of conductive material.

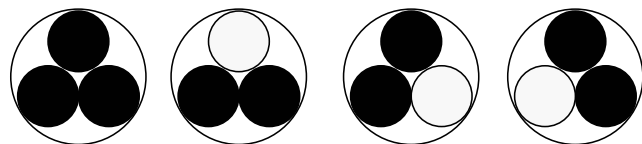
The physical problem is to derive the electrical conduction across a macroscopic region in terms of the micro-details. More generically, it is known as the percolation phenomenon (analogous to coffee flowing through a percolator).

The micro-details are described in terms of atomic lattice sites that are either occupied or unoccupied. At higher levels of description, a cluster of three units is ‘occupied’ if and only if 2 or more of the units are ‘occupied’. Think of the unit as ‘electrically conductive’ across any two points of its boundary if and only if it is ‘occupied’. Let the probability that any atomic site is occupied be  $p_0$ , where the probability of occupation at each atomic site is independent of whether any other site is occupied. This number  $p_0$  provides the exact micro-description of the system. The problem is to predict whether a piece of material of macroscopic size will be conductive or not across its boundaries.

The problem is solved as follows: Denote the probability that a cluster of three atomic sites is ‘occupied’ by  $p_1$ . Then  $p_1$  is the sum of the probabilities of the 4 mutually exclusive states depicted in Fig. 2. The probability of the first is clearly  $p_0^3$ , since the independence assumption tells us to multiply the occupation probabilities. Likewise, the second configuration has probability  $p_0^2(1-p_0)$ , as do the last two. Therefore,  $p_1 = p_0^3 + 3p_0^2(1-p_0)$ . By the same reasoning,  $p_2 = p_1^3 + 3p_1^2(1-p_1)$ , and so on. In general, if  $p$  denotes the occupation probability at level  $n$  and  $p'$  is the occupation probability at level  $n+1$ , then

$$p' = R(p) = p^3 + 3p^2(1-p).$$

The function  $R(p)$  is called the *renormalization group* (RG) transformation. The name arises from the original problems in QFT, and may be misleading. The fact that the transformation can be described as a group (by adding inverses and an identity element) plays no role.



**Figure 2:** At the atomic level, a site is either occupied or unoccupied. At any higher level, a unit is ‘occupied’ if and only if 2 or more of the 3 components are ‘occupied’. There are 4 *mutually exclusive* possibilities, above.

What is important to the problem is that the transformation has 3 fixed points. A fixed point, by definition, is a value of

<sup>1</sup> I recommend Cardy (1996) and Kadanoff (2000) as examples of physics texts that are conceptually and mathematically clearer than most, and probably all, of their predecessors. Wilson’s original papers (see references) are remarkably clear considering that they do not have the benefit of 30 years of hindsight wisdom.

$p$  that is left unchanged by the RG transformation. It is easy to verify that  $p = 0$  is a fixed point, since  $R(0) = 0$ . So, if the occupation probability at the micro-level is 0, then the probability of macroscopic conduction is 0. Similarly,  $R(1) = 1$ , which means that if the occupation probability at the atomic level is 1, then the probability of macroscopic conduction is 1. These consequences are fairly obvious, but seeing how they follow from the fixed point properties of renormalization group transformation is instructive.

There is a third fixed point at  $p = 1/2$ . However, the physical properties of this fixed point are not so obvious. It does imply that if atomic occupation probability is *exactly*  $1/2$  then the macroscopic probability of conduction is  $1/2$ . But this fixed point is unstable. It's like a ball on top of a round bowl. For any value of  $p_0$  smaller than  $1/2$ , the probability of being conductive at the macroscopic level is 0, whereas for any value of  $p_0$  above  $1/2$ , the probability of conduction is 1.  $p_c = 1/2$  is a *critical fixed point*, while 0 and 1 are stable attractors.

Some of the properties of this example are the same as in the more difficult applications of the renormalization group in physics. In particular, the macroscopic behavior of the system is governed by the fixed points of the transformation rather than the micro-details of the system. The *exact* value of  $p_0$  is irrelevant to the macro-behavior. The RG description succeeds in abstracting away from the messy micro-details.

Perhaps the most philosophically interesting property of the example is the sense in which *every* ontological level contributes to the physics. The macroscopic behavior of the system is *not* a *direct* function of its microscopic properties. Rather, the functional connection is mediated by what happens at all intermediate levels of description. This happens in the QFT examples in which every intermediate energy level contributes to the physics, and in the statistical mechanics examples, where statistical fluctuations at every wavelength count (between the atomic spacing and the correlation length).

There are two possible philosophical interpretations. One could argue that new information about physical interactions is added at each level, so each level is to some extent ontologically independent of the others. The contrary viewpoint is that the RG transformation implements a kind of iterated elimination, and ultimately everything supervenes on interactions at the micro-level. The percolation example provides an easy-to-understand battle ground for both sides. The extension of the debate to more difficult examples is then easier to follow.

## References:

- Cardy, John (1996), *Scaling and Renormalization in Statistical Physics*. Cambridge: Cambridge University Press.
- Chowdhury, Debashish and Stauffer, Dietrich (2000), *Principles of Equilibrium Statistical Mechanics*. Weinheim: Wiley-VCH.
- Kadanoff, Leo P. (2000), *Statistical Physics, Statics, Dynamics and Renormalization*. Singapore: World Scientific.
- Wilson, Kenneth G. (1971a), "Renormalization Group and Critical Phenomena: I Renormalization Group and the Kadanoff Rescaling Picture," *Physical Review B*, **4**, 3174-3183.
- Wilson, Kenneth G. (1971b), "Renormalization Group and Critical Phenomena: II Phase Cell Analysis of Critical Phenomena," *Physical Review B*, **4**, 3184-3205.
- Wilson, Kenneth G. (1983), "The Renormalization Group and Critical Phenomena," *Reviews of Modern Physics*, **55**, 583-600.
- Wilson, Kenneth G. and J. Kogut (1974), "The Renormalization Group and the  $\epsilon$  Expansion," *Physics Reports*, **12**, 75-200.