CHAPTER FIVE

Intertemporal General Equilibrium Theory

The Cambridge criticisms show that neither Clark’s nor the Austrian approach provide any warranted theory of the rate of interest. Whatever general justification neoclassical theories possess, that justification does not carry over to simplified capital theories. Economists have no reason to believe they can measure capital by time in such a way that the rate of interest is a decreasing function of the period of production. Since, holding all other factors constant, the rate of interest can increase when the quantity of capital, no matter how one tries to measure it, increases, theorists are unjustified in conceiving of interest as a payment to a factor of production. It thus appears that marginal utility theory fails to give any warranted coherent account of the nature of capital and interest.

Such a conclusion is not yet justified. Neoclassical economists have another arrow in their quiver. Samuelson (1961–62) begins with these comments:

Repeatedly in writings and lectures I have insisted that capital theory can be rigorously developed without using any Clark-like concept of aggregate “capital,” instead relying on a complete analysis of a great variety of heterogeneous physical capital goods and processes through time. (1961–62:193)

Solow seconds Samuelson on this point:

The highbrow answer is that the theory of capital is, after all, just a part of the fundamentally microeconomic theory of the allocation of resources, necessary to allow for the fact that commodities can be transformed into other commodities over time. Just as the theory of resource allocation has as its “dual” a theory of competitive pricing, so the theory of capital has as its “dual” a theory of intertemporal pricing involving rentals, interest rates, present values, and the like. In both cases, a complete price theory is also a theory of distribution among factors of production, if not among persons. (1963: 14)

According to Samuelson and Solow the Cambridge critics have been attacking parables and simplifications; their comments do not touch the rigorous theory. I shall now examine this “highbrow” answer—the theory of intertemporal pricing.

One must thus look beyond the simple parables to the more general models developed by contemporary theorists. Walras’ own models, or even their refinement in Lindahl’s work (1939, part III) will not be discussed, because both face serious criticisms (Eatwell 1975c; Collard 1973, and Jaffe 1942) and have been superseded by intertemporal general equilibrium models as developed by Arrow, Debreu, and Malinvaud. I know of no acceptable treatment of the rate of interest or profit which employs a model of a stationary general equilibrium.

1. Intertemporal General Equilibrium

In this chapter I shall largely follow Malinvaud’s lucid exposition (1972, ch. 10). Equilibrium theorists distinguish commodities both by their nature and by the time period in which they are available. Thereby they reinterpret the atemporal production equilibrium (ch. 2, §2) as an intertemporal equilibrium. “Goods” are commodities of the same nature. Malinvaud is concerned with the organization of the economy (of production, distribution, and consumption) during a time period from $t = 1$ to $t = T$. The date $t = 1$ is “now.” All the individuals involved make all their decisions now for the entire period up to $T$. Each agent thus decides on a “program” of activities during each period. These decisions, which are, of course, constrained by the initial endowment and its distribution and the technical possibilities, will determine the complete course of the economy over the time interval $(1, T)$. By distinguishing commodities according to the time period in which they are available, intertemporal equilibrium theorists are automatically embedding time and expectations in utility and production functions. Uncertainty is simply assumed away. Obviously these theorists are operating at a ferociously high level of abstraction.

At $t = 1$ agents operate with “discounted” prices. If the equilibrium

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1 See also Koopmans (1957, esp. pp. 105–26); Debreu (1959), Arrow and Hahn (1971), and Bliss (1975, ch. 3).
exchange ratio is three eggs in period 4 for one custard pie in period 2, \( p_{x,t}/p_{x,t-1} = 3 \). The double subscripts specify respectively what good the commodity is and what period it is available in. Notice that nothing has been said about what eggs will cost during the fourth time period. Agents complete all exchanges of current commodities and of titles to commodities of all other dates entirely within period 1. Imagine a commodity futures market in which exchangers have perfect information about the future. Such a market differs from the construction of the equilibrium theorists in one major respect. 'The 'price' [on real futures markets] to be paid is also specified now (it is the 'price' prevailing on the floor of the exchange), but it is understood that this price shall be paid at the delivery date, at the delivery location. This difference from the price concept which will be used here is inessential' (Debreu 1959:33). In period 1, the agents are actually carrying out exchanges (although for future commodities only contracts can be currently transferred). Prices are those (discounted) prices that are current in period 1.

The determination of economic equilibrium proceeds just as in ordinary general equilibrium theory. Each consumer has a utility function which he or she maximizes by adjusting purchases of commodities (and sales of resources or services) until the ratio of the marginal utilities of commodities (and resources and services) equals the (discounted) price ratio. Production functions or production sets specify what commodities in what quantities can be made available in each period. These production functions can most conveniently be represented as relating output in period \( k \) to input in period \( k - 1 \). Processes of production that require more than one time period are regarded as creating a series of intermediate products (like a partly built dam) at the end of each period. These production functions are homogeneous with constant returns to scale, positive first partial derivatives and (beyond a certain quantity), negative second partial derivatives. The result of perfect competition will be that the value of the marginal product of a commodity in each of its uses will be the same, and the marginal products of different commodities combined in the production of any commodity will be proportional to their prices. These conditions, plus the nonexistence of profit in profit-maximizing competitive equilibrium and the specification of the original endowment and its distribution, are sufficient for the existence of an equilibrium solution for the system. For a proof see Debreu's Theory of Value. For an intuitive explanation of the principle of such proofs, see Arrow (1968). A simple general equilibrium model is developed below in §5.

Intertemporal equilibrium models rest on the reinterpretation of the notion of a commodity. Once commodities are distinguished by the date at which they are available, atemporal production equilibria (which have no produced inputs) are transformed into intertemporal equilibria, which can have intermediate products and capital goods. With a simple reinterpretation of the notion of a commodity, models apparently without any relevance to the phenomena of capital and interest become (at least in the view of equilibrium theorists) the key to capital theory!

2. Own Rates of Return

How does interest enter? Malinvaud defines what he calls the "own rate of return" as follows:²

\[
 r_{q,t} = \frac{P_{q,t}}{P_{q,t+1}} - 1
\]

If the discounted price of a good \( q \) at \( t + 1 \) is less than the discounted price of \( q \) at \( t \), then the own rate of return of \( q \) at \( t \) is positive: people are willing to exchange one unit of \( q \) at \( t \) only for more than one unit of \( q \) at \( t + 1 \). Own rates of return express how a single good exchanges with itself over time. A good available today which exchanges for more of itself next year has a positive own rate of return this year. A good available in two years which exchanges for less of itself three years from now has a negative own rate of return two years from now.

Given the utility functions of consumers one can define a one-period own subjective interest rate for each good:

\[
 r'_{q,t} = \frac{S'_{q,t}}{S_{q,t+1}} - 1
\]

where \( S \) is the utility function and \( S'_{q,t} \) is the value of its partial derivative with respect to commodity \( q \) at \( t \). \( r'_{q,t} \) is likely to be positive for most goods because of "time preference" and the fact that consumption plans in a progressive society involve the consumption of more of \( q \) at \( t + 1 \) than at \( t \).

Malinvaud defines the (own) one-period technical interest rate as

² Malinvaud puts it differently. First he defines what he calls the "own discount factor," \( p_{q,t}/p_{q,t+1} \), and then defines the own rate of interest in terms of the own discount factor (1972:232).
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follows:

\[ r'_{q,t} = \frac{f'_{q,t}}{f'_{q,t+1}} - 1 \]

where \( f'_{q,t} \) is the value of the partial derivative of the production function with respect to commodity \( q \) at \( t \). The technical interest rate will be greater than zero if using \( x \) in production today produces more of \( x \) in the next period. Growing trees will have a positive technical interest rate, while picking strawberries may have a negative technical interest rate.

Maximization of utility insures that the ratio of the marginal utilities equals the price ratio. Maximization of profits insures that the ratio of the marginal products also equals the price ratios. In competitive equilibrium \( r_{b,t} = r_{q,t} = r_{a,t} \). Thus own rates of return reflect both subjective and objective factors of interest determination. One must again be careful in speaking of "determination." \( r_{b,t} \) or \( r_{q,t} \) or \( r_{a,t} \) or \( S_{q,t} \) are not more primitive than are \( r_{a,t} \) or \( p_{a,t} \). All of these are simultaneously determined by the given to the intertemporal equilibrium system.

3. The Rate of Interest.

Nothing, however, has been said about the rate of interest. The various own rates of return will not be equal and even if they were we would still not have all the pieces of a theory of the rate of interest, because we have not yet learned anything about contemporary prices—the prices of goods within any of the time intervals.

If the current price of tomatoes is $110 and the discounted price of tomatoes to be delivered in one year is $100, the own rate of return for tomatoes in period 1 is 10%. But this says nothing about the rate of interest or the rate of return on investing the tomatoes. The rate of interest can be any number at all larger than \(-1\). In prices actually ruling one year from now, tomatoes might be worth $330 and the rate of interest 200%, or tomatoes might be worth in next year's prices $99, in which case the rate of interest is \(-10\). Nothing said so far specifies what the rate of interest is.

In fact the general equilibrium solution is completely independent of the rate of interest and asserts nothing explicit about it. The rate of interest has no role whatsoever in the general equilibrium solution. This sounds more paradoxical than it is, since the rate of interest and contemporary prices (which also are not part of the general equilibrium solution) are dependent on one another. Economists can stipulate whatever rate of interest they want, but if their stipulation is too large, the price level will increase. A changing price level shows that the nominal interest rate specified differs from the real rate of interest. Furthermore, it can easily be shown (and will be later in this section) that whatever rate of interest is stipulated, each investment earns exactly that. The characteristic property of a capitalist economy, that in equilibrium there is an equal rate of return on all investments, is preserved.

The general equilibrium solution provides implicitly a theory of the rate of interest. Malinvaud suggests that one pick a numeraire good which is such that when its price is set equal to 1 in each period, the general price level will remain steady (1972:233, 240–41). The own rate of return on this numeraire good is (for each period and investment term) the rate of interest. There are, of course, problems in defining a suitable price index. The more the goods available in different periods differ, the more difficult it is to say whether the price level has remained steady. That the numeraire commodity in each period be the same good is only a notational convenience. To simply my discussion, I shall consider only the one-period rate of interest, \( r \). If \( Q \) is the numeraire good and \( p_{q,t} \) is the contemporary price of good \( q \) in period \( t \), \( p_{a,t} / p_{Q,t} \), then one can derive the following:

\[
\frac{1 + r_{q,t}}{1 + r} = \frac{p_{q,t}}{p_{q,t+1}} + \frac{p_{Q,t}}{p_{Q,t+1}} \text{ by definition}
\]

\[
= \frac{p_{q,t}}{p_{q,t+1}}
\]

The prices of all those commodities whose own rate of return is larger than the rate of interest will decline, while the price of all those whose own rate of return is less than the rate of interest will increase. General stability in the pricing system shows that one has picked the numeraire well and correctly determined the real rate of interest.

4. Is There Such a Thing as the Rate of Interest?

C. J. Bliss argues that thinking in terms of intertemporal equilibrium theory should lead economists to stop talking about the rate of interest altogether. "Our idea is nearly the exact antithesis of Solow's proposal.
It is that capital theory should be liberated from the concept of the rate of interest, meaning by that one rate. . . . Instead, we will find the concept of intertemporal prices to be fundamental and will see that working with the rate of interest is a clumsy groping for that concept (1975:10). Notice in fairness to Solow that he too regards highbrow theory as concerned with intertemporal pricing (1963:14). In so far as Bliss is only arguing that economic theory should be developed in terms of intertemporal prices, his claim depends on the relative adequacy of intertemporal general equilibrium theory. If such theories are indeed superior, economists should agree with Bliss that intertemporal prices are theoretically fundamental. 4

But Bliss wants, I think, to go further and to assert that there is something mistaken or incoherent in talking about a single rate of interest. Of the traditional neoclassical view that capital accumulation results in a falling rate of interest, he says:

In the first place, as we have had reason to remark and underline above, the rate of interest is not a legitimate concept outside the particular and special conditions of semi-stationary growth with a constant-rate-of-interest price system. The orthodox vision includes the statement that the rate of interest will decline as capital accumulation proceeds. Strictly, in the present case, that statement cannot be interpreted. We have a whole structure of interest rates, even in one week, not a single rate of interest. Which rate of interest should decline to validate the orthodox vision? The question is obtuse. . . . All that one can reasonably say with regard to the rate of interest is that as a description of an optimal development . . . the orthodox vision fails to make sense. (1975:294)

Two separate questions need to be considered. First, Bliss asserts that there is no single n-period rate of interest at a given time. This assertion is misleading as the informal comments above suggest and as will be proved below. 5

Frank Hahn makes the same assertion more emphatically:

For all specifications it will not be the case that the equilibrium price of a good for future delivery in terms of the same good for current delivery will be the same for all goods. But then, as all the world knows, there is no such thing as "the rate of profit." If general equilibrium analysis takes the special case of an economy with constant returns to scale and linear Engel curves, then it is easy to show that for a special set of initial conditions there will be a uniform equilibrium rate of profit. To say that a very extreme specialization of a general model somehow shows the latter to be inapplicable requires the very summit of incomprehension. (1975, p. 360)

Bliss and Hahn are exaggerating. Hahn is right that contemporary prices and the rate of interest are entirely irrelevant to decision-making at the beginning of the initial period. The rate of interest and the numeraire for each time period can be regarded as entirely arbitrary. In the basic model "there is no such thing as 'the rate of profit,'" but one should not leap to any conclusions about real economies. 6 In real economies there are, of course, a large variety of different rates of return on different financial and real assets, but neither this fact nor the possibility of setting up intertemporal models that do not mention single (n-period) rates of interest shows that there is in fact "no such thing as 'the rate of profit.'" For any real economy one can add up the values of the returns on investments after subtracting out rents, risk premiums, and depreciation that turn up nominally as profits. One can also add up the values of the various investments for a period. After correcting for price changes, since different periods are involved, one can calculate the rate of profit.

Intertemporal equilibrium theory does not say that the rate of profit is entirely arbitrary and without any systematic relation to the functioning of the economy (Malinvaud 1972:240–41). Let me first show that the rate of return is equal on all investments of the same length and then argue that it is not arbitrary. That the rate of profit is equal on all investments is a perfectly general result, a trivial implication of the zero-profit condition. I shall only prove it for one-period investments. Consider any investment in period t resulting in a commodity bundle b in period t + 1. Let the input be the vector a. Let p0 and p t+1 be row vectors of discounted prices in periods t and t + 1. Then, from

4 The "superiority" here is problematic as is the notion that intertemporal theory is more fundamental. See chapters 7, 9, and 10.
5 There may be terminological difficulties; I have been speaking indiscriminately of the rate of interest and the rate of profit, \( r_e = r_{0,1} \) is called by Malinvaud (1972:240–41) "The rate of profit." I doubt that Bliss would concede the existence of a single rate of profit.

6 Robinson (1961:57) jumps to a related, but equally misleading conclusion: "in a market economy, either there may be a tendency towards uniformity of wages and the rate of profit in different lines of production, or prices may be governed by supply and demand, but not both. Where supply and demand rule, there is no room for uniform levels of wages and the rate of profit... . . . each producer in a Walrasian system can have a prospective rate of return on investment in his own line, but there is no mechanism to equalize profits between one line and another."
the zero profit condition, one knows that

\[ p_{t+1} \cdot b = p_t \cdot a \]

Let the actual rate of return on the investment be \( r_{y,t} \). Thus

\[ \bar{p}_{t+1} \cdot b = \bar{p}_t \cdot a (1 + r_{y,t}) \]

If \( p_{Q,t} \) is the discounted price of the numeraire commodity in period \( t \) and \( p_{Q,t+1} \) is the discounted price of the numeraire commodity in period \( t + 1 \),

\[ \bar{p}_{t+1} = \frac{p_{t+1}}{p_{Q,t+1}} \text{ and } \bar{p}_t = \frac{p_t}{p_{Q,t}} \]

Substituting in the production equation containing contemporary prices,

\[ (p_{t+1} \cdot b) \frac{p_{Q,t}}{p_{Q,t+1}} = p_t \cdot a (1 + r_{y,t}) \]

Since

\[ p_{t+1} \cdot b = p_t \cdot a \text{ and } p_{Q,t}/p_{Q,t+1} = (1 + r_t), \]

\( 1 + r_t = 1 + r_{y,t} \text{ and } r_t = r_{y,t} \). The return on all investments equals the rate of interest.

Not only is the rate of profit on all investments of the same length which are made in the same period identical, but it is not arbitrary. This point was already briefly made above. The rate of interest will depend on the choice of a numeraire, but if one chooses badly, the general price level will increase or decrease. The rate of profit or interest after one corrects for inflation or deflation is determinate. There is no perfect measure of price stability, but there are usable and nonarbitrary standards. To give a simple criterion: the price level in period 2 is equal to the price level in period 1 if and only if \( \bar{p}_1 \cdot q_1 = \bar{p}_2 \cdot q_1 \) where \( \bar{p}_1 \) and \( \bar{p}_2 \) are row vectors of the contemporary prices in the two periods and \( q_1 \) is the column vector of the commodities available in period one. What this means is that the price level is roughly stable if it costs the same this year to buy the goods that one bought last year. Other, and possibly preferable, standards of price stability are available.

Bliss has, however, a second point to make, which is valid. The rates of interest between any two time periods will in general differ. To regard this proposition as a difficulty for previous theories of interest does not seem fair. Economists did not need intertemporal equilibrium models to tell them that rates of interest on investments of different lengths differ or that rates of interest in different time periods differ. Indeed the thesis of a falling rate of interest or profit is incoherent without a recognition of such differences. That interest rates thus vary does not constitute a decisive objection to theorizing in terms of a single average rate of interest on investments of different terms. Stationary state or proportional growth models in which such a single rate of interest figures are highly unrealistic. But can one do any better? If one wants to deal seriously with the term structure of interest rates, intertemporal equilibrium models are not yet a great improvement, since the crucial factor, uncertainty, is still usually assumed away.

Furthermore, Solow is right to stress the importance of the rate of return to actual entrepreneurial decisions. A theoretical construct in terms of intertemporal prices, in which the rate of return has no explicit role to play, may or may not be a valid approach. It should, however, give one some way of relating these prices to the rate of return on investments involving various risks, if the model is to be relevant to the actual operation of the economy.

An intertemporal equilibrium solution implicitly determines rates of return in each period on investments of any length. Such returns are not explicitly mentioned in the solution, but as soon as one specifies a numeraire which holds the price level constant, one can calculate the rates of return. Only in stationary or proportional growth equilibria will there be only one rate of interest for investments of any length.

5. An Intertemporal Model

One can better appreciate what intertemporal equilibrium models are and how they implicitly provide theories of capital and interest by considering a simple example. The following model is far more restrictive than it needs to be, but the restrictions add to its simplicity and make it easier to compare with the models presented in Chapters 3, 4, and 8.

\(^7\) At period \( t \) the returns on one- and two-period investments need not be equal. No forces of competition demand that they be. Competition does, however, demand that the discounted sum of the returns on a one-period investment at \( t \) and on a one-period investment at \( t + 1 \) equals the discounted return on a two-period investment at \( t \).
Assumptions of the model: I have tried to distinguish those assumptions which are peculiar to the particular model from those which are common in models of intertemporal general equilibrium.

A: Setting

A1. Time period: There are only two time periods, \( t = 1 \) and \( t = 2 \). (In the general case there can be any finite number of time periods and the analysis can even be extended to an infinite time horizon.)

A2. Consumers and owners of resources and endowments: There are \( n \) consumers, divided into two disjoint classes—owners of the initial endowments and laborers. (In general there will be no such simple division of agents.)

A3. Services, resources and endowments: At \( t = 1 \), we have the vector of original endowments \( (x_1, m_1) \) where \( x \) represents wood and \( m \) represents axes. We also have a fixed quantity of homogeneous labor, \( L^* \). \( x_2 \) and \( m_2 \) represent the commodities in period 2. All commodities, services, and resources are infinitely divisible. (In general, besides original endowments, each period may have its own vector of primary resource inputs. Labor can be heterogeneous and the vector of labor inputs from period to period may vary exogenously or as the result of production and consumption decisions. One can also specify all the possible commodities that may (but need not) exist in some period or other.)

B: Production

B1. The production functions: There are two of these, \( x_2 = f(x_1, m_1, L) \) and \( m_2 = h(x_1, m_1, L) \). Each is homogeneous of degree 1. The first partial derivatives are positive and the second partial derivatives are, after some quantity of \( x_1, m_1, \) or \( L \), negative. (In general economists specify a single implicit production function from which some explicit production functions can be derived or a set of implicit production functions, one for each period. These functions are homogeneous of degree one with positive first and, after a point, negative second partial derivatives. Not every commodity in any time period need be an input at all, let alone an input into the production of all commodities. Joint products are also perfectly acceptable. As was mentioned in chapter 2, these calculus formulations are largely outdated.)

B2. Entrepreneurs or firms aim to maximize profits.

C: Consumption

C1. Utility functions: Each consumer has a utility function that has (up to an unreached point of satiation) positive first and negative second partial derivatives. Moreover, in the specific example, I shall assume that aggregate utility functions, \( U^L \), for laborers and \( U^C \) for owners of non-labor resources are known in advance. Both commodities are consumption goods. Taking the aggregate utility functions as given in advance is equivalent to assuming that all owners of like original endowments have the same utility functions, so that the aggregate is not affected by prices or incomes. (In general, aggregate utility functions cannot be known independently of the equilibrium solution and are not theoretically useful. Not every commodity need be an argument in each utility function.)

C2. Consumers maximize utility as constrained by their initial holdings of commodities or resources.

D: Market Conditions

D1. There is no excess demand.
D2. No one is able to influence the prices of what he or she buys or sells.
D3. There is free mobility of labor and all commodity inputs.
D4. All parties on the market have complete and accurate information concerning quantities and prices of commodities and technological possibilities.
D5. All parties are perfectly creditworthy and have limitless access to credit.
D6. Total expenditure equals total income.
D7. Total expenditure equals total income in each period.
D8. All available labor is expended.
D9. The wage is paid at the beginning of \( t = 1 \). (Assumptions D7–D9 are special assumptions which are made only with respect to the special model I am developing. No use of D-5 nor mention of credit will figure in this special model.)

Assumptions of group A or of group D would not (when asserted of actual economies) be counted by anyone as laws. They are merely
simplifications needed to set up a system of equations or inequalities for which one can prove that an equilibrium solution exists. The specification of initial conditions by assumptions of groups A or D gives rise to philosophical perplexity, because many of these simplifications are not only false with respect to most modern economies, but false with respect to all known economies. As idealizations they are numerous and extreme. Assumptions B and C, on the other hand, seem to involve lawlike claims. These issues will be discussed in chapters 6 and 7.

The Intertemporal Model

On the basis of the assumptions above, one can set up the following system of equations that have an equilibrium solution:

\[ x_2 = f(x_1, m_1, L) \]  
\[ m_2 = h(x_1, m_1, L_1) \]  
\[ L^x + L^m = L^* \]

where the superscript indicates the good into which the labor is an input.

Maximization of profits (B2) plus the features of production functions (B1) and some calculus manipulations insure that the marginal products of inputs bear the same ratio to one another as do their prices.  

\[ \frac{\overset{*}{f}_{m_1}}{\overset{*}{f}_{x_1}} = \frac{p_{m_1}}{p_{x_1}} \]  
\[ \frac{\overset{*}{f}_{L}}{\overset{*}{f}_{x_1}} = \frac{w}{p_{x_1}} \]  
\[ \frac{\overset{*}{h}_{m_1}}{\overset{*}{h}_{x_1}} = \frac{p_{m_1}}{p_{x_1}} \]  
\[ \frac{\overset{*}{h}_{L}}{\overset{*}{h}_{x_1}} = \frac{w}{p_{x_1}} \]

Furthermore, market clearing (D1) implies:

\[ m_1 = m_1^L + m_1^C + m_1^x + m_1^m \]  
\[ x_1 = x_1^L + x_1^C + x_1^x + x_1^m \]

Since the income of workers must equal their expenditures (D6 and denying D5),

\[ wL^* = p_{x_1} \cdot x_1^L + p_{m_1} \cdot m_1^L \]

Finally zero profit in competitive equilibrium (which follows from f and h being homogeneous of degree one and D2–4, 7) requires that

\[ p_{x_2} \cdot x_2 = p_{x_1} \cdot x_1^x + p_{m_1} \cdot m_1^x + wL^x \]  
\[ p_{m_2} \cdot m_2 = p_{x_1} \cdot x_1^m + p_{m_1} \cdot m_1^m + wL^m \]

There are 16 independent equations and 17 unknowns \((x_1^x, x_1^m, x_1^L, x_1^C, x_2, m_1^x, m_1^m, m_1^L, m_1^C, m_2, L^*, L'^*, p_{x_1}, p_{x_2}, p_{m_1}, p_{m_2}, w)\). Taking \(p_{x_1} = 1\), the system is perfectly determinate. The general existence proof for intertemporal equilibrium systems proves that this system is solvable.

\[ \frac{\overset{*}{U}_{m_1}}{\overset{*}{U}_{x_1}} = \frac{p_{m_1}}{p_{x_1}} \]  
\[ \frac{\overset{*}{U}_{m_2}}{\overset{*}{U}_{x_1}} = \frac{p_{m_2}}{p_{x_1}} \]  
\[ \frac{\overset{*}{U}_{C_1}}{\overset{*}{U}_{x_1}} = \frac{p_{x_2}}{p_{x_1}} \]  
\[ \frac{\overset{*}{U}_{C_2}}{\overset{*}{U}_{x_1}} = \frac{p_{x_2}}{p_{x_1}} \]
The two own rates of return are by definition:

\( r_{x,1} = \frac{p_{x,1}}{p_{x,2}} - 1 \)  \hspace{1cm} (5.17)

\( r_{m,1} = \frac{p_{m,1}}{p_{m,2}} - 1 \)  \hspace{1cm} (5.18)

Since the actual rate of return, whatever it is, must be equal on all investments

\( r_{x,1} = \frac{p_{x,1}}{p_{x,2}} - 1 = \frac{p_{x,1}(1 + r_1)}{\bar{p}_{x,2}} - 1 \)  \hspace{1cm} (5.19)

\( r_{m,1} = \frac{p_{m,1}}{p_{m,2}} - 1 = \frac{p_{m,1}(1 + r_1)}{\bar{p}_{m,2}} - 1 \)  \hspace{1cm} (5.20)

If \( r_1 \) is larger than both \( r_{x,1} \) and \( r_{m,1} \), \( \bar{p}_{x,2} \) is larger than \( p_{x,1} \) and \( \bar{p}_{m,2} \) is larger than \( p_{m,1} \). If \( r_1 \) is less than both of the own rates of return, the price of both wood and axes will decline. If \( r_1 \) is the real rate of interest in period 1, then it must be between or equal to \( r_{x,1} \) or \( r_{m,1} \).

The model presented in this section, despite its extreme simplicity, exemplifies the structure of intertemporal models generally. Given preference structures, production possibilities, the initial endowment, and its distribution, one proves that there exists at least one intertemporal equilibrium for the economy. In such a proof, one derives discounted prices, outputs, discounted incomes, consumption, actual technology employed, and the distribution of income. All of these quantities are independent of contemporary prices or the rate of interest. Given a numeraire for each period, one can derive contemporary prices and the nominal rate of interest. If one chooses the numeraires so that the price level remains steady, the nominal rate of interest will equal the real rate of interest.

6. Austrian and Intertemporal Models

Abstract intertemporal general equilibrium models help one to appreciate the relations among various economic models. Austrian models of capital and interest can be regarded as models of intertemporal general equilibrium to which additional restrictions have been added. Most previous economic models can be regarded as special versions of intertemporal equilibrium models. To convert the simple model of §5 into Lange's model, one need only specify that the contemporary prices of goods are unchanging, and that output and its distribution is the same in all periods and equal to the initial endowment and its distribution and give \( x \) no role in production. The resulting model is still more powerful than Lange's, since the utility functions provide additional information. They would make it possible to determine the value of Lange's \( k \) and to 'explain' what limits money capital and restricts production from reaching its maximum. Dropping the utility functions, one has Lange's model. Lange's model, like the model of §5, is an equilibrium model to which particular consistent additional constraints have been added. In Lange's model, some information has also been omitted.

The two major contributions that arise from Böhm-Bawerk's work are the two 'laws': (3.21) more roundabout processes increase the output from the services of land and labor at a diminishing rate; and (3.22) agents prefer present to future consumption. These generalizations need not be part of intertemporal equilibrium models in general, although in fact they are partly embodied in almost all intertemporal models. In reaching an intertemporal general equilibrium constrained by (nonaggregate) production functions and utility functions, it need not be the case that own technical and subjective interest rates are positive for all commodities. Prices of individual commodities can change over time. The weighted average must, however, be positive, if there is a positive rate of return. In the model of §5, for example, it need not be the case that the marginal product of an ax in ax making be larger than one ax or that the capitalist prefer present axes to axes in period 2. The own rate of interest for axes may be negative. In that case, speaking loosely, wood must follow Böhm-Bawerk's 'laws' or else the rate of return for the whole economy would be negative. Since \( x \) and \( m \) are inputs into the production of both goods, there is no general way to measure capital or time intensity. Some of the relations between time, utilities, and production that the Austrians were looking for at the aggregate level are preserved in certain averages and at the level of individual commodities; but no simple conclusions concerning aggregate capital, an average period of production, or the rate of interest can be reached. Only by further restrictions does it become possible to speak of a period of production and to achieve a simplified model of capital and interest.

\* I shall argue in chapter 7 that intertemporal general equilibrium models ought not themselves to be regarded as the fundamental theoretical constructions of neo-classical theory. The basis of neo-classical theory lies in what I shall call 'equilibrium theory.'
What distinguish Austrian models of capital and interest from equilibrium models in general are these further restrictions. These restrictions include not only Böhm-Bawerk’s two basic insights, but also simplifications which permit one to state these insights as generalizations concerning “the average period of production” and “the preference for present commodities” and “the rate of interest.” Three simplifications are essential: (1) the analysis is confined to stationary equilibria; (2) consumer goods are not inputs into production, and (3) the time structure of production permits calculation of the time intensity. Models which exemplify or embody the Austrian theory of capital and interest are models which are simplified in this way and which behave in accordance with Böhm-Bawerk’s two generalizations.

We can now understand better how the Cambridge criticisms bear on the Austrian theory and on the whole neoclassical approach to capital and interest. The Cambridge counter-models can be regarded as special case general equilibrium models. None of the assumptions of the Cambridge models is inconsistent with those of the most abstract intertemporal general equilibrium model. The reader can verify this claim by comparing the Cambridge counter-models of chapter 4 with the most general form of the assumptions of intertemporal equilibrium models discussed in the last section. The point will be much more obvious after we consider the basic conceptual structure of neoclassical economics in chapter 6. If the assumptions of the Cambridge models are, as I have asserted, consistent with those of abstract general equilibrium models, it follows immediately that the Cambridge economists have not produced even hypothetical counter-examples to abstract general equilibrium theories of capital and interest.

The difficulties the Cambridge critics have identified lie entirely with the addition to intertemporal equilibrium models of Böhm-Bawerk’s lawlike assumptions and the three simplifications listed above. In adding these five assumptions, one is going beyond what general equilibrium theories warrant. The Cambridge critics have not revealed any flaw in general equilibrium models, nor have they shown that Böhm-Bawerk’s lawlike assumptions ought not to be partly embodied in them. Even general equilibrium models which assume that in equilibrium own technical and subjective interest rates are positive give one no reason to believe that there is any simple relation between interest and the value of capital or the time intensity of production. Consider Bliss’ comment:

Even people who have made no study of economic theory are familiar with the idea that when something is more plentiful its price will be lower,